LAGRANGE REMAINDER OR ERROR BOUND

Like alternating series, there is a way to tell how accurately your Taylor polynomial approximates the actual function value: you use something called the **Lagrange remainder** or **Lagrange error bound**.

Lagrange Remainder: If you use a Taylor polynomial of degree n centered about c to approximate the value x, then the actual function value falls within the error bound

$$R_n(x) = \frac{f^{(n+1)}(z)\,(x-c)^{n+1}}{(n+1)!} \quad , \quad \text{where z is some number between x and c.}$$

Translation: Similar to alternating series, the error bound is given by the next term in the series, n + 1. the only tricky part is that you evaluate $f^{(n+1)}$, the (n + 1)th derivative, at z, not c. z is the number that makes $f^{(n+1)}(z)$ as large as it can be. This error bound is supposed to tell you how far off you are from the real number, so we want to assume the worst. We want the error bound to represent the largest possible error. In practice, picking z is pretty easy.

Example 1:

Approximate cos (.1) using a fourth-degree Maclaurin polynomial, and find the associated Lagrange remainder (error bound).

Solution:

Since the 4th degree Taylor polynomial for $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, then

$$\cos(.1) \approx 1 - \frac{(.1)^2}{2!} + \frac{(.1)^4}{4!} \approx \boxed{.99500416667}$$

Now, the associated Lagrange remainder after n = 4 (denoted $R_4(x)$) is

$$R_4(x) = \frac{f^{(5)}(z)\,(x-c)^5}{5!} \quad .$$

The fifth derivative of cos x is –sin z. Now, plug in x = .1 and c = 0 to get

$$R_4(.1) = \frac{(-\sin z)(.1)^5}{5!} \quad .$$

We need – sin z to be as large as possible. The largest value of – sin z is 1. By assuming – sin z is the largest possible value, we are creating the largest possible error; so, plug in 1 for – sin z. The actual remainder will be less that this largest possible value.

$$R_4(.1) < \frac{(1)\,(.1)^5}{5!} = \frac{.1^5}{5!} = \boxed{.0000000833}$$

Therefore, our approximation of .99500416667 is off by less than .0000000833.

Example 2:

(a) Determine the degree of the Maclaurin polynomial that should be used to approximate $\sqrt[3]{e}$ to four decimal places. (b) Use this Maclaurin polynomial to estimate $\sqrt[3]{e}$ to four decimal places.

Solution:

(a) $f(x) = e^x$ The n^{th} degree Maclaurin polynomial is $P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!}$

The Lagrange form of the remainder with $x = \frac{1}{3}$ $\left(\text{since } \sqrt[3]{e} = e^{1/3}\right)$ is

$$R_n\left(\frac{1}{3}\right) = \frac{f^{(n+1)}(z)\left(\frac{1}{3}\right)^{n+1}}{(n+1)!} \quad \text{where } 0 < z < \frac{1}{3}$$

Since $f^{(n)}(x) = e^x$ for all derivatives of $f(x) = e^x$, we have

$$\left|R_n\left(\frac{1}{3}\right)\right| < \frac{e^{1/3}}{(n+1)!}\left(\frac{1}{3}\right)^{n+1}$$

$$\left|R_n\left(\frac{1}{3}\right)\right| < \frac{e^{1/3}}{(n+1)!\,3^{n+1}}$$

but since $e < 27$, then $e^{1/3} < 3$ and we have:

$$\left|R_n\left(\frac{1}{3}\right)\right| < \frac{3}{(n+1)!\,3^{n+1}}$$

$$\left|R_n\left(\frac{1}{3}\right)\right| < \frac{1}{(n+1)!\,3^{n}} \qquad \text{(the Lagrange error bound)}$$

Since we are seeking $\sqrt[3]{e}$ with four decimal accuracy, we need $\left|R_n\left(\frac{1}{3}\right)\right|$ to be less than 0.00005.

So, $\left|R_n\left(\frac{1}{3}\right)\right| < 0.00005$ when $\frac{1}{(n+1)!\,3^{n}} < 0.00005$

By trial and error using a calculator, this is true when $n = 5$ since $\frac{1}{(5+1)!\,3^5} \approx .000006 < .00005$

Therefore, we use $\boxed{P_5\left(\frac{1}{3}\right)}$ as an approximated value of $\sqrt[3]{e}$ accurate to 4 decimal places.

(b) Then $P_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$

So $P_5\left(\frac{1}{3}\right) = 1 + \left(\frac{1}{3}\right) + \frac{(1/3)^2}{2!} + \frac{(1/3)^3}{3!} + \frac{(1/3)^4}{4!} + \frac{(1/3)^5}{5!} = \frac{5087}{3645} \approx 1.39561$

Therefore, $\sqrt[3]{e} \approx 1.3956$ accurate to 4 decimal places.