Kepler's 3rd Law Derivation

Kepler never did this derivation. He jumped straight to the result in 1609 by pouring over astronomical data. At the time of Kepler, Calculus didn't exist. Newton invented Calculus largely using astronomically known quantities and used Calculus to prove Kepler's equation. This derivation utilizes the Lagrangian to simplify Newton's proof.

Part 1.

Velocity squared in polar coordinates: $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$

$\dot{r}^2$ is the velocity-squared component inward, while $r^2\dot{\theta}^2$ is the velocity-square component along the direction of motion.

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

Part 2.

$$L = T - U$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

For an object in orbit, the angular momentum doesn't change as the object moves throughout its orbit:

$$\frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U$$

Potential energy only depends on radial distance from the source, so $\frac{\partial U}{\partial \dot{\theta}} = 0$.

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m \frac{\partial}{\partial \dot{\theta}} \left(r^2 \dot{\theta}^2 \right)$$

$$\frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

Since $\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = 0$, $\left(m r^2 \dot{\theta} \right)$ is constant.

$$\boldsymbol{m r^2 \dot{\theta} \equiv l}$$

l is an arbitrary constant used to define a very useful quantity for future steps. $l = m r^2 \dot{\theta}$

$$\frac{l}{m} = r^2 \dot{\theta}$$

$$\frac{l^2}{m^2} = r^4 \dot{\theta}^2$$

The following is a useful upcoming substitution:

$$\frac{l^2}{m^2 r^2} = r^2 \dot{\theta}^2$$

Part 3.

$$E = T + U$$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + U$$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\left(\frac{l^2}{m^2r^2}\right) + U$$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2} + U$$

$$\frac{1}{2}m\dot{r}^2 = E - U - \frac{1}{2}\frac{l^2}{mr^2}$$

$$\dot{r} = \pm\sqrt{\frac{2}{m}(E-U) - \frac{l^2}{m^2r^2}}$$

$$d\theta = d\theta\frac{dt}{dt}\frac{dr}{dr} = \frac{d\theta}{dt}\frac{dt}{dr}dr$$

$$d\theta = \frac{\dot{\theta}}{\dot{r}}dr$$

$$\theta = \int \pm\frac{m\dot{\theta}}{\sqrt{2m\left(E - U - \frac{l^2}{2mr^2}\right)}}dr$$

Make that same substitution in the numerator:

$$\theta = \int \pm\frac{l/r^2}{\sqrt{2m\left(E - U - \frac{l^2}{2mr^2}\right)}}dr$$

Part 4.

$$F = \frac{Gm_1m_2}{r^2}$$

$$U = \int F \cdot dr = \int \frac{Gm_1m_2}{r^2} = -\frac{Gm_1m_2}{r}$$

$$U = -\frac{k}{r} \quad , \quad k \equiv Gm_1m_2$$

$$\theta = \int \pm \frac{l/r^2}{\sqrt{2m\left(E + \frac{k}{r} - \frac{l^2}{2mr^2}\right)}} dr$$

Use u-substitution with $u \equiv \frac{l}{r}$:

$$\theta = \int \frac{u^2}{\sqrt{au^2 + bu + c}}$$

Use integral tables and many steps...

$$\cos\theta = \frac{\frac{l^2}{mk} \cdot \frac{1}{r} - 1}{\sqrt{1 + \frac{2El^2}{mk^2}}}$$

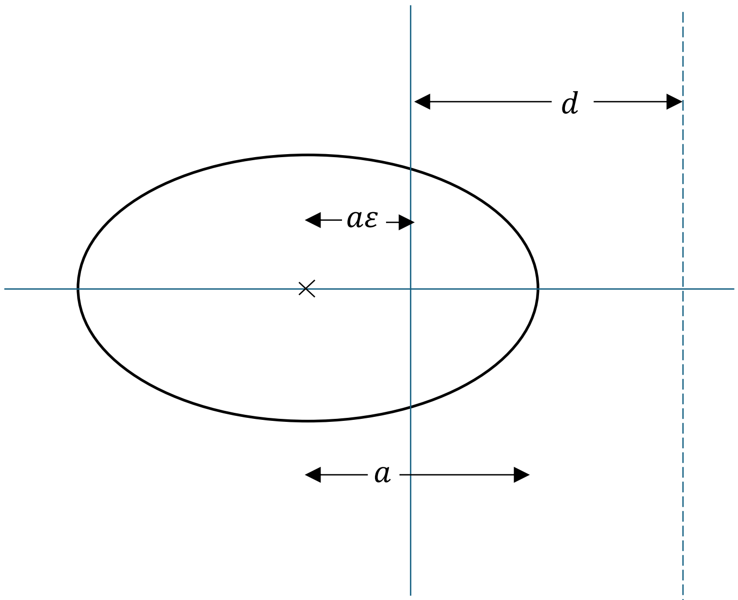
Create two definitions:

$$\alpha \equiv \frac{l^2}{mk} \quad , \quad \varepsilon \equiv \sqrt{1 + \frac{2El^2}{mk^2}}$$

$$\cos\theta = \frac{\alpha \cdot \frac{1}{r} - 1}{\varepsilon}$$

$$\boldsymbol{\frac{\alpha}{r} = 1 + \varepsilon \cos\theta}$$

Part 5.



The geometry of a conic section:

$$r = \frac{\varepsilon d}{1 + \cos\theta}$$

This becomes an ellipse if $0 < \varepsilon < 1$.

If, $\alpha \equiv \varepsilon d$, the orbital motion equations for a central force are ellipses!!!

The derivation for the motion of an orbit with a central force happens to give an ellipse.

Part 6.

$$a \equiv semimajor\ axis$$

$$b \equiv semiminor\ axis$$

From geometry:

$$a = \frac{\alpha}{1-\varepsilon^2}$$

$$b = \frac{\alpha}{\sqrt{1-\varepsilon^2}}$$

$$\frac{\alpha}{1-\varepsilon^2} = \frac{l^2/mk}{1-\sqrt{1+\frac{2El^2}{mk^2}}^2} = \frac{l^2/mk}{2El^2/mk^2} = \frac{k}{2E}$$

$$\frac{\alpha}{\sqrt{1-\varepsilon^2}} = \frac{l^2/mk}{\sqrt{1-\sqrt{1+\frac{2El^2}{mk^2}}^2}} = \frac{l^2/mk}{\sqrt{2El^2/mk^2}} = \frac{l}{\sqrt{2mE}}$$

$$\boldsymbol{a = \frac{k}{2E}} \quad , \quad \boldsymbol{b = \frac{l}{\sqrt{2mE}}}$$

Part 7.

From Kepler's 2nd Law:

$$dA = \frac{1}{2} r^2 d\theta$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

Remember, $r^2\dot{\theta} = \frac{l}{m}$ so, $\frac{1}{r^2\dot{\theta}} = \frac{m}{l}$

$$dt = \frac{2m}{l} dA$$

$$\int dt = \int \frac{2m}{l} dA$$

$$T = \frac{2m}{l} A$$

Area of an ellipse = πab

$$T = \frac{2m}{l} (\pi ab)$$

$$T = \frac{2m}{l} \pi \frac{k}{2E} \cdot \frac{l}{\sqrt{2mE}}$$

$$T = \frac{\pi k \sqrt{\frac{m}{2}}}{\sqrt{E^3}}$$

$$a = \frac{k}{2E} \quad \therefore \quad E = \frac{k}{2a}$$

$$T = \frac{\pi k \sqrt{\frac{m}{2}}}{\sqrt{\left(\frac{k}{2a}\right)^3}}$$

$$T^2 = \frac{\pi^2 k^2 \frac{m}{2}}{\frac{k^3}{8a^3}} = \frac{4\pi^2 m a^3}{k}$$

$$k \equiv G m_1 m_2$$

$$\boldsymbol{T^2 = \frac{4\pi^2 a^3}{Gm}}$$